

The Complexity of All (g, f) -Factor Problem

Hongliang Lu *

School of Mathematics and Statistics

Xi'an Jiaotong University

Xi'an, Shaanxi 710049, China

February 21, 2017

Abstract

Let G be a graph with vertex set V and let $g, f : V \rightarrow \mathbb{Z}^+$ such that $g \leq f$. We will say that G has all (g, f) -factors if G has an h -factor for every $h : V \rightarrow \mathbb{Z}^+$ such that $g(v) \leq h(v) \leq f(v)$ for every $v \in V$ and $h(V(G)) \equiv 0 \pmod{2}$. Niessen (A characterization of graphs having all (g, f) -Factors, *J. Combin. Theory, Ser. B*, **72** (1998), 152–156) derived from Tutte's f -factor theorem a similar characterization for the property of having all (g, f) -factors and asked whether there is a polynomial algorithm for testing whether a graph G has all (g, f) -factors. In this paper, we show that it is NP-hard to determine whether a graph G has all (g, f) -factors, which gives a negative answer for this question of Niessen.

1 Introduction

We consider finite simple graphs, which have neither loops nor multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$.

Let $w(G)$ denote the number of components of a graph G . A graph G is t -tough if $|S| \leq tw(G - S)$ for every subset S of the vertex set $V(G)$ with $w(G - S) > 1$. The *toughness* of G , denoted $t(G)$, is the maximum value of

*luhongliang@mail.xjtu.edu.cn; Supported by the National Natural Science Foundation of China under grant No.11471257 and Fundamental Research Funds for the Central Universities

t for which G is t -tough (taking $t(K_n) = \infty$ for all $n \leq 1$). We say that a graph G is *almost 1-tough* if for any $S \subseteq V(G)$,

$$w(G - S) \leq |S| + 1.$$

Let $H : V(G) \rightarrow Z^+$ be a set function. An H -factor F is a spanning subgraph such that $d_F(v) \in H(v)$ for every $v \in V(G)$. Given two integer-valued function $g, f : V(G) \rightarrow Z^+$ such that $g(v) \leq f(v)$ for all $v \in V(G)$, an H -factor is also called (g, f) -factor if $H(v) = [g(v), f(v)]$ for all $v \in V(G)$. In particular, if $g = f$, then an (f, f) -factor is also called an f -factor. We call that G has all (g, f) -factors if for any integer-valued function defined $V(G)$ such that $h(V(G)) \equiv 0 \pmod{2}$ and $g(v) \leq h(v) \leq f(v)$ for all $v \in V(G)$, G contains an h -factor. We write

$$H^{-1}(1) := \{v \in V(G) : H(v) = \{1\}\}.$$

For simplicity, we write $g^{-1}(1) := H^{-1}(1)$ if $H(v) = \{g(v)\}$ for all $v \in V(G)$.

Niessen [7] derived from Tutte's f -factor theorem a similar characterization for the property of having all (g, f) -factors.

Theorem 1 *G has all (g, f) -factors if and only if*

$$g(D) - f(S) + d_{G-D}(S) - q_G(D, S, g, f) \geq \begin{cases} -1, & g \neq f; \\ 0, & g = f. \end{cases}$$

for all disjoint sets $D, S \subseteq V$, where $q_G(D, S, g, f)$ denotes the number of components C of $G - (D \cup S)$ such that there exists a vertex $v \in V(C)$ with $g(v) < f(v)$ or $e_G(V(C), S) + f(V(C)) \equiv 1 \pmod{2}$.

It is well-known that there exists a polynomial algorithm to determine whether a graph has a (g, f) -factor. An open problem was naturally proposed in [7]:

Problem 2 *Is there a polynomial algorithm for testing whether a graph G has all (g, f) -factors?*

In this note, we obtain the following result, which gives a negative answer for Problem 2.

Theorem 3 *It is NP-hard to determine whether a graph G has all (g, f) -factors.*

For completing the proof of Theorem 3, we need the following two results.

Theorem 4 (Bauer et.al., [3]) *It is NP-hard to recognize 1-tough cubic graphs.*

Theorem 5 (Kano and Lu, [6]) *Let G be a connected graph. A graph G has an H -factor for every $H : V(G) \rightarrow \{\{1\}, \{0, 2, 4, \dots\}\}$ with $|H^{-1}(1)|$ even if and only if G is almost 1-tough.*

Theorem 5 gives a polynomial reduction from toughness to degree constrained factors.

2 The Proof of Theorem 3

For completing our proof, we need the following notations.

Definition 6 *Let G be a graph with $V(G) = \{v_1, \dots, v_n\}$. Write $X = \{x_1, \dots, x_n\}$, $Y = \{y_1, \dots, y_n\}$.*

- (i) *For any vertex x of G , let G^x denote the graph obtained from G by adding a new vertex x' together with a new edge xx' , that is, $G^x = G + xx'$.*
- (ii) *Let G_L be a graph with vertex set $V(G_L)$ and edge set $E(G_L)$, where $V(G_L) \cup X \cup Y$ and $E(G_L) = E(G) \cup \{x_i y_i, y_i v_i, v_i x_i \mid i \in [n]\}$;*
- (iii) *For any $H : V(G) \rightarrow \{\{1\}, \{0, 2\}\}$, we define $h_H : V(G_L) \rightarrow \{1, 2\}$ such that*

$$h_H(u) = \begin{cases} 2, & \text{if } H(u) = \{0, 2\} \text{ and } u \in V(G); \\ 1, & \text{otherwise.} \end{cases}$$

Now we show the following result.

Lemma 7 *A graph G is 1-tough if and only if for any $x \in V(G)$, G^x is almost 1-tough.*

Proof. Necessity. Suppose that G is 1-tough. Let $x \in V(G)$. For any $S \subseteq V(G^x)$,

$$w(G^x - S) \leq w(G - S) + |\{x'\}| \leq |S| + 1.$$

Sufficiency. By contradiction, suppose that G is not 1-tough. Then there exists $\emptyset \neq S \subseteq V(G)$ such that $w(G - S) \geq |S| + 1$. Let $x \in S$. One can see that

$$w(G^x - S) = w(G - S) + 1 \geq |S| + 2,$$

contradicting to that G is 1-tough. \square

From Theorem 4 and Lemma 7, one may see that

Lemma 8 *It is NP-hard to recognize almost 1-tough cubic graph.*

Lemma 9 *Let G be a connected cubic graph and let $H : V(G) \rightarrow \{\{1\}, \{0, 2\}\}$. G contains an H -factor if and only if G_L contains an h_H -factor.*

Proof. Necessity. Suppose that G contains an H -factor F . Define $M_1 = \{x_i y_i \mid d_F(v_i) \in \{1, 2\} \text{ for } i \in [n]\}$ and $M_2 = \{x_i v_i, y_i v_i \mid d_F(v_i) = 0 \text{ for } i \in [n]\}$. Let F' be a spanning subgraph of G_L with edge set $E(F) \cup M_1 \cup M_2$. From the definition of function h_H , one can see that

$$h_H(x) = d_{F'}(x) = \begin{cases} 2, & \text{if } H(x) = \{0, 2\} \text{ and } x \in V(G); \\ 1, & \text{otherwise.} \end{cases}$$

So F' is an h_H -factor of G_L .

Sufficiency. Suppose that F' be an h_H -factor of G_L . Let F be a spanning subgraph of G with edge set $E(F') \cap E(G)$. Now we show that F is an H -factor of G . From Definition 6 (iii), one can see that

$$H(u) = \begin{cases} \{1\}, & h_H(u) = 1 \text{ and } u \in V(G); \\ \{0, 2\}, & h_H(u) = 2 \text{ and } u \in V(G). \end{cases}$$

Consider $h_H(v_i) = d_{F'}(v_i) = 1$. Since $d_{F'}(x_i) = d_{F'}(y_i) = 1$, then we have $x_i v_i, y_i v_i \notin E(F')$. So we have $d_F(x_i) = 1 \in H(v_i)$. Next we may assume that $d_{F'}(v_i) = 2$. Then either $\{x_i v_i, y_i v_i\} \subseteq E(F')$ or $\{x_i v_i, y_i v_i\} \cap E(F') = \emptyset$. In the former case, we have $d_F(v_i) = 0$, and in the latter case we have $d_F(v_i) = 2$. So in both cases, one can see that $d_F(v_i) \in \{0, 2\}$ and F is an H -factor of G . This completes the proof. \square

Lemma 10 *Let G be a connected cubic graph. G contains an H -factor for any $H : V(G) \rightarrow \{\{1\}, \{0, 2\}\}$ with $|H^{-1}(1)|$ even if and only if G_L contains all (g, f) -factors, where $g \equiv 1$ and*

$$f(u) = \begin{cases} 2, & \text{if } u \in V(G); \\ 1, & \text{else } u \in V(G_L) - V(G). \end{cases}$$

Proof. Define

$$\mathcal{H} = \{H \mid H : V(G) \rightarrow \{\{1\}, \{0, 2\}\} \text{ such that } |H^{-1}(1)| \text{ is even}\}.$$

and

$$\mathcal{F} = \{h \mid h : V(G_L) \rightarrow \{1, 2\} \text{ such that } |h^{-1}(1)| \text{ is even and } h(v) = 1 \text{ for every } v \in X \cup Y\}.$$

Let $J : \mathcal{H} \rightarrow \mathcal{F}$ such that $J(H) = h_H$ for all $H \in \mathcal{H}$. From the definition of h_H , one can see that J is well-defined. By Lemma 9, it is enough for us

to show that J is a bijection function. Firstly, we show that J is injective. For any $H, H' \in \mathcal{H}$ such that $H \neq H'$, there exists $x \in V(G)$, such that $H(x) \neq H'(x)$. Without loss generality, we may assume that $H(x) = \{1\}$ and $H'(x) = \{0, 2\}$. By Definition 6 (iii), we have $h_H(x) = 1$ and $h_{H'}(x) = 2$, which implies $h_H \neq h_{H'}$. Next we show J is a surjection. For every $h \in \mathcal{F}$, we may define $H : V(G) \rightarrow \{\{1\}, \{0, 2\}\}$ such that

$$H(v) = \begin{cases} \{1\}, & \text{if } h(v) = 1 \text{ and } v \in V(G); \\ \{0, 2\}, & \text{else } h(v) = 2 \text{ and } v \in V(G). \end{cases}$$

Since $|h^{-1}(1)|$ is even, $|H^{-1}(1)| = |h^{-1}(1) \cap V(G)|$ is even. So we infer that $H \in \mathcal{H}$ and $J(H) = h$. Thus J is surjective.

This completes the proof. \square

Proof of Theorem 3. Let G be a connected cubic graph. By Theomre 5, G is almost 1-tough if and only if G contains an H -factor for any $H : V(G) \rightarrow \{\{1\}, \{0, 2\}\}$ with $|H^{-1}(1)|$ even. Thus by Lemma 10, one can see that G is almost 1-tough if and only if G_L contains all (g, f) -factors, where $g \equiv 1$ and

$$f(u) = \begin{cases} 2, & \text{if } u \in V(G); \\ 1, & \text{else } u \in V(G_L) - V(G). \end{cases}$$

Since it is NP-hard to recognize almost 1-tough grphs by Lemma 8, one can see that it is NP-hard to determine whether a graph contains all (g, f) -factors. This completes the proof of Theorem 3. \square

References

- [1] D. Bauer, S.L. Hakimi and E. Schmeichel, Recognizing tough graphs is NP-hard, *Discrete Appl. Math.*, **28** (1990), 191–195.
- [2] D. Bauer, H.J. Broersma and H.J. Veldman, Not every 2-tough graph is hamiltonian, *Discrete Appl. Math.*, **99** (2000), 317–321.
- [3] D. Bauer, J. van den Heuvel, A. Morgana and E. Schmeichel, The complexity of recognizing tough cubic graphs, *Discrete Appl. Math.*, **79** (1997), 35–44.
- [4] D. Bauer, J. van den Heuvel, A. Morgana and E. Schmeichel, Toughness and triangle-free graphs. *J Combin. Theory Ser. B*, **65** (1995), 208–221.
- [5] G. Cornuéjols, General factors of graphs, *J. Combin. Theory Ser. B*, **45** (1988), 185–198.

- [6] M. Kano and H. Lu, Characterization of 1-Tough Graphs using Factors, preprint.
- [7] Niessen, A characterization of graphs having all (g, f) -Factors, *J. Combin. Theory, Ser. B*, **72** (1998), 152–156.
- [8] W.T. Tutte, The factorization of linear graphs, *J. London Math. Soc.*, **22** (1947), 107–111.